Speaker: Zinovy Reichstein (University of British Columbia)

Title: Essential dimension: a survey

Abstract: Let K/k be a field extension. The essential dimension $\operatorname{ed}(\alpha)$ of an algebraic object α (e.g., a quadratic form, a finite-dimensional algebra, an algebraic variety, a group action, etc.) defined over K is the minimal value of $\operatorname{trdeg}_k(K_0)$ such that α descends to K_0. Here K_0 ranges over the intermediate fields $k \subset K_0 \subset K$.

Let G be an algebraic group defined over k. The minimal value of $\operatorname{ed}(\alpha)$, as α ranges over all G-torsors over $T \to \operatorname{Spec}(K)$ and K ranges over the field extensions of k, is called the essential dimension of G and is denoted by $\operatorname{ed}(G)$. This numerical invariant of the algebraic group G naturally arises in a number of contexts. To the best of my knowledge, this numerical invariant first appeared (in a special case) in the 1884 book *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom 5ten Grade* by Felix Klein. In our terminology, Klein showed that the essential dimension of the symmetric group S_5 (viewed as a finite constant group over $k = \mathbb{C}$) is 2. The problem of computing the essential dimension of the symmetric group S_n, which remains open to this day for every $n \geq 7$, is related to the algebraic form of Hilbert's 13th problem.

The groups of essential dimension zero are the so-called special groups, introduced by Serre and classified by Grothendieck (over an algebraically closed field) in the 1950s. The problem of computing the essential dimension of a general algebraic group may be viewed as a natural extension of this theory. This problem has attracted a great deal of attention over the past ten years, with substantial progress achieved over the past two. In these lectures, based on joint work with P. Brosnan, J. Buhler, Ph. Gille, A. Vistoli and B. Youssin, I will survey the known results on this problem and the methods used to obtain them. I will also discuss the related notion of canonical dimension and other related topics.